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PROPAGATION OF TORSIONAL SURFACE WAVES IN A HOMOGENEOUS SUBSTRATUM OVER A HETEROGENEOUS HALF-SPACE

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SUMMARY

The paper discusses the propagation of torsional surface wave in a homogeneous substratum over a half-space with linearly varying rigidity and density. The study reveals that under assumed conditions, a torsional surface wave propagates in the medium. The velocities of torsional surface waves have been calculated numerically and are presented in a number of graphs. It is also observed that for a stratum over a homogeneous half-space, the velocity of torsional surface waves coincides with that of Love waves. For a non-homogeneous half-space it is observed that the velocity of torsional surface waves is always higher than that of Love waves propagated in a homogeneous layer over a homogeneous half-space. An attempt is also made to assess the possible propagation of torsional surface waves in a half-space with linearly varying rigidity and density, lacking a superficial layer. It is concluded that such a half-space allows two solutions for torsional waves while a homogeneous half-space has one.

KEY WORDS: torsional surface waves; propagation; heterogeneous half-space

INTRODUCTION

The study of surface waves in elastic media is most important to seismologists due to its possible applications in geophysical prospecting and in understanding the causes of damage due to earthquakes. Quite a good amount of information about the propagation of seismic waves is available in the well-known book by Ewing *et al.*[1] A large number of papers on the subject have been published in various journals. In fact, the study of surface waves for homogeneous, non-homogeneous and layered media has been a central interest to theoretical seismologists until recently. Of these, the commendable works by Vrettose[2,3] on surface waves in inhomogeneous medium may be cited. His study give much insight on the effects of non-homogeneity on surface waves causes by line loads although much information is available on the propagation of surface waves, such as Rayleigh waves, Love waves and Stonel waves, etc.; torsional waves have not drawn much attention and only scant literature is available on the propagation of such waves.

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Lord Rayleigh[4] in his remarkable paper showed that an isotropic homogeneous elastic half-space does not allow torsional surface waves to propagate. Later on, Meissner[5] pointed out that in an inhomogeneous elastic half space with quadratic variation of shear modulus and density varying linearly with depth, torsional surface waves do exist. Recently, Vardoulakis[6] has shown that torsional surface waves also propagate in a Gibson half-space, that is, a half-space in which the shear modulus varies linearly with depth but the density remains unchanged. Torsional waves in an initially stressed cylinder have been studied by Dey and Dutta[7] and the existence and propagation of torsional surface waves in an elastic half-space with void pores has been discussed by Dey *et al.*[8]

This paper presents a study on the propagation of torsional surface waves in a homogeneous layer of finite thickness over a heterogeneous half-space. Both the rigidity and density of the half-space are assumed to vary linearly with depth. It is observed that such a medium allows torsional waves to propagate. Prominent effect of variation of density on the propagation of torsional wave is observed in the low-frequency range. As a particular case, the velocity of torsional surface waves in a non-homogeneous half-space in the absence of superficial layer is also discussed, and it is found that two torsional wave fronts are possible in the low-frequency range. For a homogeneous layer over a homogeneous half-space, it is observed that the speed of torsional waves coincides with that of Love waves.

FORMULATION

Consider a homogeneous layer of thickness H, over a vertically heterogeneous half-space. The heterogeneity has been considered both in density as well as rigidity.

Considering the origin of the cylindrical co-ordinate system at the interface of the layer and the z-axis downward positive, the following variation in rigidity and density is taken:

(i) for the layer, $\mu = \mu_0$, $\rho = \rho_0$
(ii) for the half-space, $\mu = \mu_1(1 + az)$, $\rho = \rho_1(1 + bz)$.

where μ and ρ are the rigidity and the density of the media, respectively, and a, b are constants having dimensions that are inverse of length.

EQUATION OF MOTION

For torsional wave propagation in the radial direction, the equation of motion may be written as

$$\frac{\partial}{\partial r}\sigma_{r\theta} + \frac{\partial}{\partial r}\sigma_{z\theta} + \frac{2}{r}\sigma_{r\theta} = \rho(z)\frac{\partial^2 v}{\partial t^2} \tag{1}$$

with $v(r, z, t)$ being the displacement along the θ (azimuthal) direction. For an elastic medium the stresses are related to the displacement component v by

$$\sigma_{r\theta} = \mu(z)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$$

$$\sigma_{z\theta} = \mu(z)\left(\frac{\partial v}{\partial z}\right) \tag{2}$$

Using equation (2), equation (1) takes the form

$$\mu(z)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)v+\frac{\partial}{\partial z}\left[\mu(z)\frac{\partial v}{\partial z}\right]=\rho(z)\frac{\partial^2 v}{\partial t^2} \tag{3}$$

We assume a solution of (3) of the form

$$v = V(z)\ J_1(Kr)\,e^{iwt} \tag{4}$$

where $V(z)$ is the solution to

$$V''(z)+\frac{\mu'(z)}{\mu(z)}V'(z)-K^2\left(1-\frac{C_1^2}{C_s^2}\right)V(z)=0 \tag{5}$$

in which $C_1 = w/K$ and $C_s = (\mu/\rho)^{1/2}$ and $J_1(Kr)$ is the Bessel function of first kind.

SOLUTION FOR THE UPPER LAYER

For the upper layer

$$\mu = \mu_0 \quad \text{and} \quad \rho = \rho_0 \tag{6}$$

Using (6) and (5), the equation for the upper layer takes the form

$$V''(z)-K^2\left(1-\frac{C_1^2}{C_0^2}\right)V(z)=0 \tag{7}$$

where

$$C_0=\left(\frac{\mu_0}{\rho_0}\right)^{1/2} \tag{8}$$

The solution of (7) is

$$V(z)=A_1\exp\{(1-C_1^2/C_0^2)^{1/2}Kz\}+A_2\exp\{-(1-C_1^2/C_0^2)^{1/2}Kz\}, \tag{9}$$

and hence the displacement in the upper homogeneous layer is

$$v=[A_1\exp\{(1-C_1^2/C_0^2)^{1/2}Kz\}+A_2\exp\{-(1-C_1^2/C_0^2)^{1/2}Kz\}]J_1(Kr)\,e^{iwt} \tag{10}$$

SOLUTION FOR THE HALF-SPACE

For the half-space

$$\begin{aligned}\mu &= \mu_1(1+az)\\ \rho &= \rho_1(1+bz)\end{aligned} \tag{11}$$

and (5) takes the form

$$V''(z)+\frac{a}{(1+az)}V'(z)-K^2\left[1-\frac{C_1^2(1+bz)}{C_2^2(1+az)}\right]V(z)=0 \tag{12}$$

where

$$C_2 = \left(\frac{\mu_1}{\rho_1}\right)^{1/2}$$

Substituting $V(z) = \phi(z)/(1 + az)^{1/2}$, equation (12) takes the form

$$\phi''(z) + \left[\frac{a^2}{4(1 + az)^2} - K^2 \left\{1 - \frac{C_1^2(1 + bz)}{C_2^2(1 + az)}\right\}\right] \phi(z) = 0 \tag{13}$$

Using the dimensionless quantities $\gamma = (1 - C_1^2\, b/C_2^2\, a)^{1/2}$ and $\eta = 2\gamma K(1 + az)/a$ in equation (13), one gets

$$\phi''(\eta) + \left[\frac{P}{2\eta} - \frac{1}{4} + \frac{1}{4\eta^2}\right] \phi(\eta) = 0 \tag{14}$$

where

$$P = \frac{w^2(a - b)}{C_2^2\, a^2 \gamma\, K}$$

The solution of equation (14) satisfying the condition $\lim_{z \to \infty} V(z) \to 0$, i.e. $\lim_{\eta \to \infty} \phi(\eta) \to 0$ may be taken as

$$\phi(\eta) = D_1\, W_{p/2,0}(\eta) \tag{15}$$

where $W_{p/2,0}(\eta)$ is the Whittaker function.

Hence, the displacement component v_1 in the heterogeneous half-space is given by

$$v = \frac{D_1\, W_{p/2,0}\; [2\gamma k(1 + az)/a]}{(1 + az)^{1/2}}\, J_1(Kr)\, e^{iwt} \tag{16}$$

BOUNDARY CONDITIONS

The following boundary conditions must be satisfied:

(i) v and $\mu\,(\partial v/\partial z)$ are continous at $z = 0$,
(ii) $\mu\,(\partial v/\partial z)$ vanishes at $z = -H$.

Expansion of the Whittaker function up to the linear terms, and substituting into the set (i) of boundary conditions gives

$$A_1 + A_2 = D_1 \left(\frac{2\gamma K}{a}\right)^{1/2} e^{-\gamma K/a} \left[1 + \frac{2\gamma K A}{a}\right] \tag{17}$$

and

$$\mu_0 K \left(1 - \frac{C_1^2}{C_0^2}\right)^{1/2} [A_1 - A_2] = A_1 \mu_1 \left[2\gamma K A - \gamma K \left\{1 + \frac{2\gamma K A}{a}\right\}\right] \left(\frac{2\gamma K}{a}\right)^{1/2} e^{-\gamma K/a} \tag{18}$$

where

$$A = \frac{(1 - P)}{2}$$

The boundary condition (ii) takes the form

$$A_1 \exp\{(1 - C_1^2/C_0^2)^{1/2} KH\} + A_2 \exp\{-(1 - C_1^2/C_0^2)^{1/2} KH\} = 0 \tag{19}$$

Elimination A_1, A_2 and D_1 from (17), (18) and (19) one gets

$$\frac{\mu_1 \gamma}{\mu_0}\left[1 - \frac{2A}{(1 + 2\gamma K A/a)}\right]\frac{1}{(C_1^2/C_0^2 - 1)^{1/2}} = \tan\left\{KH\left(\frac{C_1^2}{C_0^2} - 1\right)^{1/2}\right\} \tag{20}$$

This gives the velocity of torsional surface waves in a homogeneous layer over a vertically heterogeneous half-space.

PARTICULAR CASES

Case I. When $b \to 0$, i.e. if the half-space has constant density, then, $\gamma = 1$, and $P = C_1^2 K/C_2^2 a$. Using these results in (20), the velocity equation takes the form

$$\frac{\mu_1}{\mu_0}\left[1 - \frac{2A}{(1 + 2 K A/a)}\right]\frac{1}{(C_1^2/C_0^2 - 1)^{1/2}} = \tan\left\{KH\left(\frac{C_1^2}{C_0^2} - 1\right)^{1/2}\right\} \tag{21}$$

Case II. When $a \to 0$, $b \to 0$, i.e. the layer and the half-space have both constant density and rigidity, equation (20) takes the form

$$\frac{\mu_1}{\mu_0}\frac{(1 - C_1^2/C_2^2)^{1/2}}{(C_1^2/C_0^2 - 1)^{1/2}} = \tan\{KH\,(C_1^2/C_0^2 - 1)^{1/2}\} \tag{22}$$

This is the well-known equation for Love waves in a homogeneous layer over a homogeneous half-space. This shows that in a layered isotropic homogeneous medium, torsional surface waves change asymtotically to Love waves as the distance from the origin increases.

Case III. In the absence of the upper layer ($H \to 0$), equation (20) then takes the form

$$\gamma\left[1 - \frac{2A}{(1 + 2\gamma K A/a)}\right] = 0 \tag{23}$$

This gives either

$$\frac{C_1}{C_2} = \left(\frac{a}{b}\right)^{1/2} \tag{24}$$

or

$$\left[1 - \frac{2A}{(1 + 2\gamma K A/a)}\right] = 0 \tag{25}$$

Equation (25) takes then the form

$$X_1 C^3 + X_2 C^2 + X_3 C + 1 = 0 \tag{26}$$

where

$$C=\left(\frac{C_1}{C_2}\right)^{1/2}$$

$$X_1=\frac{K^2b}{a^2}\left(1-\frac{b}{a}\right)^2$$

$$X_2=\left(\frac{2b}{a}-1\right)^2-\frac{K^2}{a^2}\left(1-\frac{b}{a}\right)^2 \tag{27}$$

$$X_3=2\left(1-\frac{2b}{a}\right)$$

Equation (26) does not give any real positive value for C when $G^2+4H_1^3>0$,

$$\text{where}\quad G=X_1^2-\frac{X_1X_2X_3}{3}+2\left(\frac{X_2}{3}\right)^3$$

$$H_1=\frac{X_1X_3}{3}-\left(\frac{X_2}{3}\right)^2$$

and hence the velocity of torsional waves is $C_1=(a/b)^{1/2}C_2$, i.e. as obtained in equation (24).

When $G^2+4H_1^3<0$, equation (26) gives positive real values of C. To find the real positive roots of C from equation (26), the values of $(K/a)^2$ are chosen so that these satisfy the condition $G^2-4H_1^3<0$ and the real roots of equation (26) are computed using the Newton–Raphson method. The results are presented in Figure 1. Hence, in the case $G^2+4H_1^3<0$, there are three wave fronts, two given by equation (26) and the third by equation (24).

When the variation with depth of the parameters for μ and ρ in the half-space are same, i.e. $a=b$ then in the absence of superficial layer the velocity of torsional wave C_1 becomes equal to C_2, implying thereby that the torsional wave velocity coincides with that of shear wave.

When $b=0$, but $a\neq 0$, then the half-space is referred to as a Gibson's half-space and the velocity of torsional wave in such a soil is given by

$$C_1=C_2\left(\frac{1}{K/a-1}\right)^{1/2}$$

which was obtained by Vardoulakis.[6]

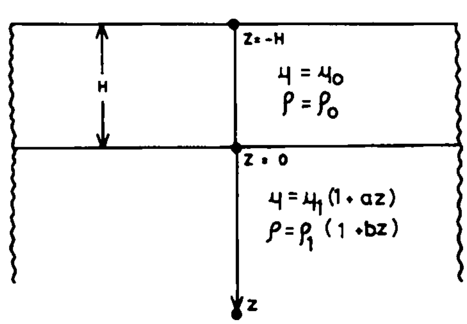

Figure 1. Geometry of the problem

When $a \to 0$, $b \to 0$, i.e. for the homogeneous half-space, from equation (23), it is found that torsional waves do not propagate in such a medium. This result was shown by Rayleigh.[4]

EXAMPLE OF APPLICATIONS AND DISCUSSION OF THE RESULTS

The values of C_1/C_0 have been computed from equation (20) for $a/K = 0{\cdot}01$ and $b/K = 0{\cdot}001$ for different values of KH. These values are shown in Figure 2 and are compared with the case where the density of the half-space is constant (i.e. $b = 0$). Curves 1 and 2 in Figure 2 show that for small values of KH, the variation in density has considerable effect on the velocity of propagation of torsional surface waves. To compare the propagation of torsional waves with that of Love waves, the velocities of Love waves have also been computed from equation (22) and the results presented as curve no. 3 in Figure 2. The results show that velocities of the torsional waves in an inhomogeneous medium are always higher than that of Love waves in a homogeneous medium. In the absence of the superficial layer over the non-homogeneous half-space, the velocity equation of the torsional surface waves are given by (24) and (26). Attempts were also made to determine the real positive roots of the cubic equation (26). Equation (24) gives constant velocity depending on the ratio of rigidity and density parameters. It is observed that for small values of K/a, equation (26) gives three real roots for $(C_1/C_2)^2$, one of which is negative. Neglecting this negative root, the other two were computed by the Newton–Raphson method and results are plotted and presented in curves 1 and 2 of Figure 3. From these results it may be concluded that for small values of K/a there are three torsional wave fronts propagating in the non-homogeneous half-space, two of which are given in Figure 3 and the third is given by equation (24).

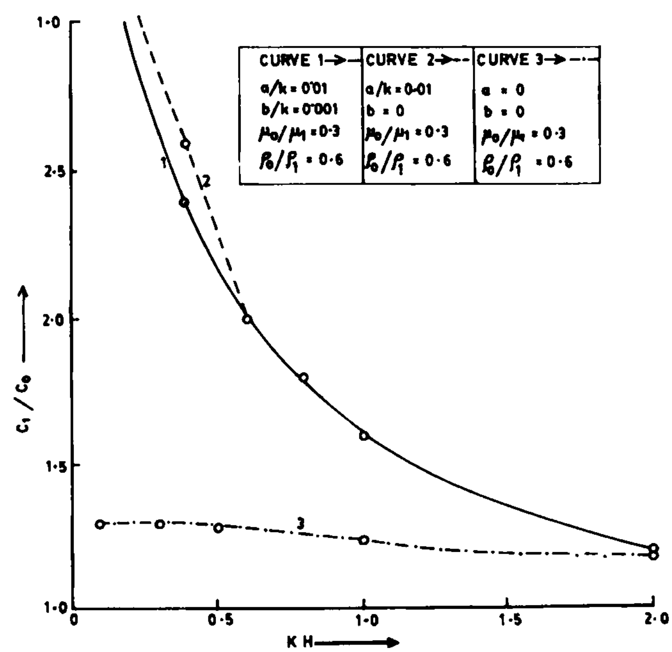

Figure 2. Torsional surface wave dispersion curves in homogeneous layer over heterogeneous and homogeneous half-space

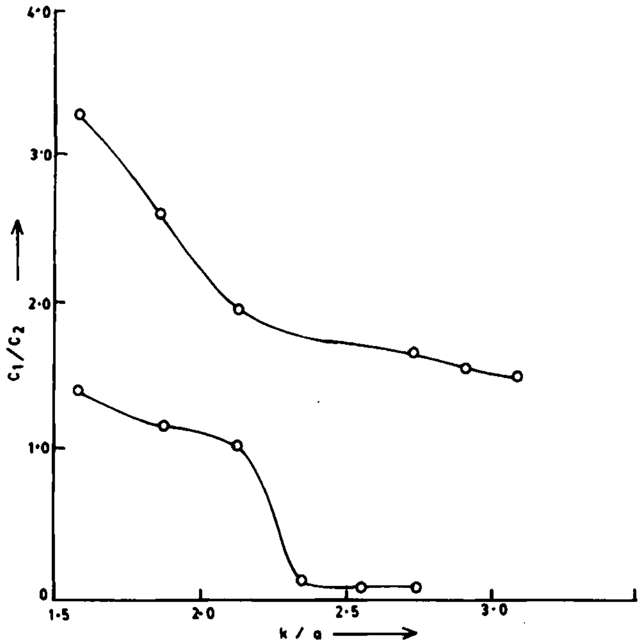

Figure 3. Torsional surface wave dispersion curve for linear variation of rigidity and density in absence of superficial layer

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